

# A Note on “H-Cordial Graphs”

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## Abstract

The concept of an H-cordial graph is introduced by I. Cahit in 1996 (Bulletin of the ICA). But that paper has some gaps and invalid statements. We try to prove the statements whose proofs in Cahit’s paper have problems, and also we give counterexamples for the wrong statements. We prove necessary and sufficient conditions for H-cordiality of complete graphs and wheels and  $H_2$ -cordiality of wheels, which are wrongly claimed in Cahit’s paper.

## 1 Introduction

H-cordial graphs is introduced by I. Cahit in [1], and as he claims they can be useful to construct Hadamard matrices since any  $n \times n$  Hadamard matrix gives an H-cordial labeling for the complete bipartite graph  $K_{n,n}$ . But of course the inverse is not necessarily true. Unfortunately Cahit’s paper has many wrong statements and proofs. For example the second part of “Lemma 2.3” obviously is not true. To see that consider trees in Figure 1.

Or the definition of a zero-M-cordial labeling there is not valid, since no such labelings exist. Here we try to recover that paper by fixing some wrong

proofs, and restating some statements. In this section we mention some definitions and preliminaries which are referred to throughout the paper.

We consider simple graphs (which are finite, undirected, with no loops or multiple edges). For the necessary definitions and notation we refer the reader to standard texts, such as [2].

For a **labeling** of a graph  $G$  we mean a map  $f$  which assigns to each edge of  $G$  an element of  $\{-1, +1\}$ . If a labeling  $f$  is given for a graph  $G$ , for each vertex  $v$  of  $G$  we define  $f(v)$  to be the sum of the labels of all edges having  $v$  as an endpoint. In other words  $f(v) = \sum_{e \in I(v)} f(e)$ , where  $I(v)$  is the set of all edges incident to  $v$ . For an integer  $c$  we define  $e_f(c)$  to be the number of edges having label  $c$ , and similarly  $v_f(c)$  is the number of vertices having the label  $c$ . The following lemma which states a simple but essential relation is immediate.

**Lemma 1.** *If  $f$  is an assignment of integer numbers to the edges and vertices of a given graph  $G$  such that for each vertex  $v$ ,  $f(v) = \sum_{e \in I(v)} f(e)$ , then  $\sum_{v \in V(G)} f(v) = 2 \sum_{e \in E(G)} f(e)$ .*

**Definition 1.** *A labeling  $f$  of a graph  $G$  is called **H-cordial**, if there exists a positive constant  $K$ , such that for each vertex  $v$ ,  $|f(v)| = K$ , and the following two conditions are satisfied,*

$$|e_f(1) - e_f(-1)| \leq 1 \quad \text{and} \quad |v_f(K) - v_f(-K)| \leq 1.$$

*A graph  $G$  is called to be **H-cordial**, if it admits an H-cordial labeling.*

The following lemma provides the most-used technique of the present paper.

**Lemma 2.** *If a graph  $G$  with  $n$  vertices and  $m$  edges is H-cordial then  $m - n$  is even.*

**Proof.** Since  $|v_f(-1) - v_f(1)| \leq 1$ , if  $n$  is even we have  $v_f(-1) = v_f(1)$ , and by Lemma 1 we have  $\sum_{e \in E(G)} f(e) = 0$ . This implies that  $m$  is even. Using a similar argument one can prove that if  $m$  is even then  $n$  is also even.  $\square$

If  $G$  is a tree,  $m - n = -1$ , so we have the following.

**Corollary 1.** *No H-cordial tree exists.*

## 2 Trees

Now that an H-cordial tree do not exist, we can study semi-H-cordiality of trees instead, which is a weaker condition than H-cordiality.

**Definition 2.** A labeling  $f$  of a tree  $T$  is called **semi-H-cordial**, if for each vertex  $v$ ,  $|f(v)| \leq 1$ , and  $|e_f(1) - e_f(-1)| \leq 1$ , and  $|v_f(1) - v_f(-1)| \leq 1$ . A tree  $T$  is called to be **semi-H-cordial**, if it admits a semi-H-cordial labeling.

In [1] “Lemma 2.3” states that if  $T$  is a tree such that each of its vertices has odd degree, then  $n_I$ , the number of internal vertices of  $T$  satisfies the following

$$n_I \equiv \begin{cases} 0 \pmod{2} & ; \text{if } n \equiv 2 \pmod{4} \\ 1 \pmod{2} & ; \text{if } n \equiv 0 \pmod{4} \end{cases}$$

We mentioned in the last section that this statement is not true. One can see this by two simple examples.

Each vertex in each of trees in Figure 1 has odd degree. The tree on the left has six vertices and one internal vertex, and the right one has eight vertices and two internal ones.

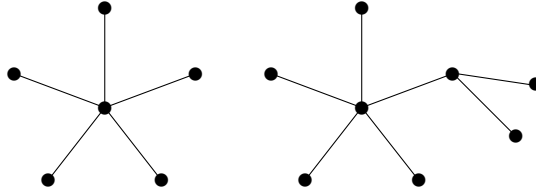


Figure 1: Counterexamples for “Lemma 2.3” of Cahit’s paper

The proofs of Lemma 2.1, Lemma 2.2, and Theorem 2.5 in [1] have serious problems, for example in some of them “Lemma 2.3” is used which we showed that is not valid. But the statements of Lemma 2.1, Lemma 2.2, and Theorem 2.5 are true and we prove all of these in the following theorem.

**Theorem 1.** A tree  $T$  is semi-H-cordial, if and only if it has an odd number of vertices.

**Proof.** Suppose that  $T$  has an even number of vertices, and  $f$  is a semi-H-cordial labeling for  $T$ . For each vertex  $v$ , we have  $f(v) \in \{-1, 0, 1\}$ , so if  $\deg v$  is even then  $f(v) = 0$ . Since  $T$  has an even number of odd vertices by a similar argument as in the proof of Lemma 2, this means that  $T$  has an even number of edges which contradicts the hypothesis.

Now assume that  $T$  has an odd number of vertices. We find a semi-H-cordial labeling  $f$  for  $T$  using an algorithm.

**Algorithm.** Define two variables  $S$  and  $a$ , where  $S$  is a set and  $a$  is a number. Initially we have  $S = E(T)$  and  $a = 1$ . Update  $S$  and  $a$  using the following two steps while  $S \neq \emptyset$ .

1. Suppose that  $e_1, e_2, \dots, e_p$  is the longest path in  $S$ . For each  $1 \leq i \leq p$  define  $f(e_i)$  to be  $(-1)^i a$  and then delete  $e_i$  from  $S$ .
2. If  $\sum_{e \in E(T) \setminus S} f(e) \neq 0$  then set  $a$  to be equal to it, otherwise set  $a = 1$ .

We claim that  $f$  is a semi-H-cordial labeling for  $T$ . First note that after each execution of the two operations, we have  $a \in \{-1, 1\}$ , because in the  $k$ -th execution if  $p$  is even, then  $f(e_1) + \dots + f(e_p) = 0$  and  $\sum_{e \in E(T) \setminus S} f(e)$  do not change. Otherwise we have  $f(e_1) + \dots + f(e_p) = -a$  and  $\sum_{e \in E(T) \setminus S} f(e)$  changes to 0 or  $-a$ . So for each  $e \in E(T)$  we have  $f(e) \in \{-1, 1\}$ . Now if the edges incident to  $v$  are completely deleted from  $S$  in the  $k$ -th execution, then we have  $f(v) = 0$  before the  $k$ -th execution and  $|f(v)| \leq 1$  after the  $k$ -th execution till the end of algorithm. On the other hand we see that  $\sum_{v \in V(T)} f(v) = 2 \sum_{e \in E(T)} f(e) = 0$ , so  $v_f(-1) = v_f(1)$ .  $\square$

In Lemma 2.6 of [1] a special case of the following proposition is stated but the proof in [1] has problem. We prove the statement in a rather simple way.

**Proposition 1.** *Let  $T$  be a tree with an even number of vertices. There exists an edge-labeling  $f$  of  $T$ , such that  $|e_f(-1) - e_f(1)| = 1$ ,  $|f(v)| \leq 1$  for each vertex  $v$  in  $T$ , and  $|v_f(-1) - v_f(1)| = 2$ .*

**Proof.** Suppose that  $v$  is a leaf in  $T$ . Add a new vertex  $w$  and a new edge  $vw$  to  $T$ . The resulting tree has a semi-H-cordial labeling  $f$  by Theorem 1 and the restriction of  $f$  to  $T$  is what we look for.  $\square$

“Theorem 2.8” [1] states that a tree is semi-H-cordial if and only if it has an even number of vertices. We have proved the opposite in Theorem 1.

### 3 H-cordial graphs

The concept of a zero-M-cordial labeling defined in [1] is useful while one tries to find an H-cordial labeling for a given graph. There are some wrongs on the concept occurred in [1]. For example the definition of a zero-M-cordial labeling given in [1] is not useful, because no such labelings exist! But the following is what one expects for a zero-M-cordial labeling.

**Definition 3.** A labeling  $f$  of a graph  $G$  is called **zero-M-cordial**, if for each vertex  $v$ ,  $f(v) = 0$ . A graph  $G$  is called to be **zero-M-cordial**, if it admits a zero-M-cordial labeling.

In [1] the definition has an additional condition  $|e_f(-1) - e_f(1)| \geq 1$ . However Lemma 1 for a zero-M-cordial labeling  $f$ , implies that  $\sum f(e) = 0$ , hence  $e_f(-1) = e_f(1)$ . So no graph may have a zero-M-cordial labeling in sense of [1].

The usefulness of the above definition appears when one tries to find an H-cordial labeling for a given graph  $G$ . If  $H$  is a zero-M-cordial subgraph of  $G$ , then H-cordiality of  $G \setminus E(H)$  simply implies H-cordiality of  $G$ . We will do so in the proof of Theorem 3.

It is immediate from the definition that a graph is zero-M-cordial, if and only if each of its components is a zero-M-cordial graph. In the following theorem we give a characterization of connected zero-M-cordial graphs.

**Theorem 2.** A connected graph  $G$  is zero-M-cordial if and only if it is Eulerian and it has an even number of edges.

**Proof.** Obviously each vertex in a zero-M-cordial graph must have even degree, and because  $e_f(1) = e_f(-1)$ , it must have an even number of edges. On the other hand if  $G$  is an Eulerian graph with even number of edges, one can label edges in order on an Eulerian tour, alternately by  $+1$  and  $-1$ , to attain a zero-M-cordial labeling.  $\square$

“Theorem 3.1” [1] gives a necessary condition for a labeling  $f$  of a connected graph  $G$ , to be H-cordial, that is the number of vertices labeled  $-1$  must be even. We show this is not always true by an example. The graph shown in Figure 2 is our example. For an H-cordial labeling of this graph, one can assign  $-1$  to thin edges and  $+1$  to thick ones.

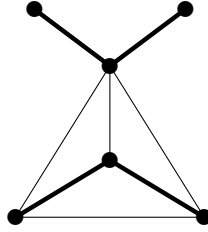


Figure 2: Counterexample for “Theorem 3.1” of Cahit’s paper

In “Theorem 3.6” [1] it is claimed that if  $n \equiv 0 \pmod{8}$ , then the complete graph  $K_n$  has an H-cordial labeling  $f$  such that  $|f(v)| = n - 1, \forall v \in V(G)$ . This is not true since if such  $f$  exists, then all edges incident with a specified vertex must have the same label. This implies that all edges of  $K_n$  must have the same label, which is impossible by the definition of an H-cordial labeling.

Theorem 3.7 [1] states that the complete graph  $K_n$  is H-cordial for  $n \equiv 0 \pmod{4}$ , and in the proof it is claimed that these are all possible H-cordial complete graphs. We show in the following theorem that this is not true.

**Theorem 3.** *A graph  $K_n$  is H-cordial if and only if  $n \equiv 0, 3 \pmod{4}$ , and  $n \neq 3$ .*

**Proof.** If  $n \equiv 1, 2 \pmod{4}$  a graph  $K_n$  can not be H-cordial by Lemma 2. Now if  $n \not\equiv 1, 2 \pmod{4}$ , we find an H-cordial labeling for  $K_n$ . We know

that if  $n$  is even, one can decompose  $K_n$  into a 1-factor and an Eulerian tour. Now if  $n \equiv 0 \pmod{4}$ , the Eulerian tour is zero-M-cordial and the 1-factor is H-cordial, so  $K_n$  is H-cordial.

Now we consider the case  $n \equiv 3 \pmod{4}$ . It is obvious that  $K_3$  has an H-cordial labeling. Suppose that  $n \geq 7$  and  $u, v, w$  are three vertices of  $K_n$ . We can find an H-cordial labeling  $f$  for  $K_n \setminus \{u, v, w\}$  as in the previous paragraph. The vertices of  $K_n \setminus \{u, v, w\}$  are  $n-3$  in number and  $(n-3)/2$  of them have label  $+1$ . So because  $p = (n-3)/4$  is an integer, we can partition  $V(K_n) \setminus \{u, v, w\}$  into  $p$  disjoint subsets  $\{a_i, b_i, c_i, d_i\}$ ,  $i = 1, \dots, p$  such that  $f(a_i) = f(b_i) = 1$  and  $f(c_i) = f(d_i) = -1$ ,  $\forall i$ . We consider two cases to complete the proof.

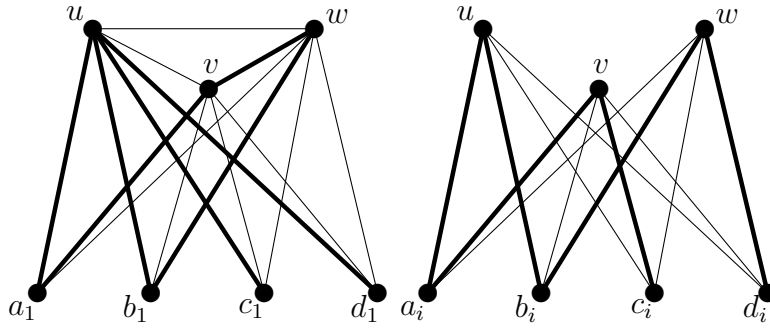


Figure 3: Labeling of  $K_n$  when  $n \equiv 3 \pmod{4}$

If  $p$  is even, we label the un-labeled edges of  $K_n$  as in Figure 3(right) where a thick edge means  $(-1)^i$  and a thin one means  $(-1)^{i+1}$ ; for the edges between  $u, v$ , and  $w$  we give to two of them label 1 and to the other one label  $-1$ . If  $p$  is odd, for  $i = 1$  we use the labels in Figure 3(left) and for  $i \geq 2$  we use the labels in Figure 3(right) where a thick edge means  $(-1)^i$  and a thin one means  $(-1)^{i+1}$ .  $\square$

“Theorem 3.9” [1] states that every cubic H-cordial graph is Hamiltonian. This is not true and the graph shown in Figure 4 is the counterexample (for a thin edge we assign the label  $-1$  and each thick edge takes  $+1$ ).

“Theorem 3.10” [1] states that the wheel  $W_n$  is H-cordial if and only if  $n \equiv 1 \pmod{4}$ . In the following theorem we show that this is not true by giving a necessary and sufficient condition for H-cordiality of a wheel.

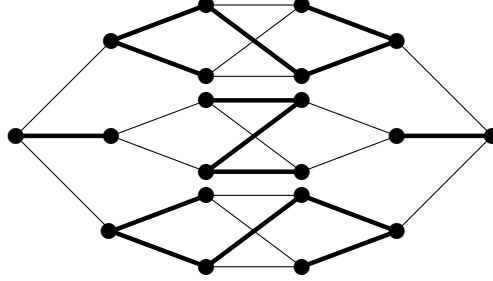


Figure 4: A cubic non-Hamiltonian H-cordial graph

**Theorem 4.** *The wheel  $W_n$  is H-cordial if and only if  $n$  is odd.*

**Proof.** If  $n$  is even, then  $W_n$  is not H-cordial by Lemma 2. On the other hand if  $n$  is odd we give an H-cordial labeling  $f$  for  $W_n$ . Suppose that  $V(W_n) = \{v_0, v_1, \dots, v_n\}$  and  $\deg v_0 = n$ . We define  $f(v_0v_i) = f(v_iv_{i+1}) = 1$  if  $1 \leq i \leq n$  and  $i$  is even, also we define  $f(v_0v_1) = 1$  and for all other edges we give the label  $-1$ . It can easily be seen that  $e_f(-1) = e_f(1) = n$ , and  $v_f(-1) = v_f(1) = (n+1)/2$ . In fact  $f(v_i) = 1$  for even  $i$ , and  $f(v_i) = -1$  for odd  $i$ .  $\square$

## 4 Generalizations

In this section we study another type of graph labeling, called  $H_k$ -cordial labeling.

**Definition 4.** *An assignment  $f$  of integer labels to the edges of a graph  $G$  is called to be a  $H_k$ -cordial labeling, if for each edge  $e$  and each vertex  $v$  of  $G$  we have  $1 \leq |f(e)| \leq k$  and  $1 \leq |f(v)| \leq k$ , and for each  $i$  with  $1 \leq i \leq k$ , we have  $|e_f(i) - e_f(-i)| \leq 1$  and  $|v_f(i) - v_f(-i)| \leq 1$ . A graph  $G$  is called to be  $H_k$ -cordial, if it admits a  $H_k$ -cordial labeling.*

The following lemma gives a necessary condition for  $H_2$ -cordiality of a graph.

**Lemma 3.** *If a graph with an even number of vertices is  $H_2$ -cordial then the number of its edges is also even.*



**Proof.** If  $f$  is a  $H_2$ -cordial labeling for a graph  $G$  and  $|V(G)|$  is even, then by lemma 1 we have  $2(e_f(1) - e_f(-1) + 2e_f(2) - 2e_f(-2)) = v_f(1) - v_f(-1) + 2v_f(2) - 2v_f(-2)$ . So  $v_f(1) = v_f(-1)$  and since  $|V(G)|$  is even,  $v_f(2) = v_f(-2)$ . Now  $e_f(1) - e_f(-1) + 2e_f(2) - 2e_f(-2) = 0$ , so  $e_f(1) - e_f(-1) = e_f(2) - e_f(-2) = 0$ .  $\square$

The converse of the above lemma is not necessarily true. A counterexample is given in Figure 5. Note that in place of bold triangle and quadruple one can put a  $C_r$  and a  $C_{r+1}$  respectively for each  $r \geq 3$ .

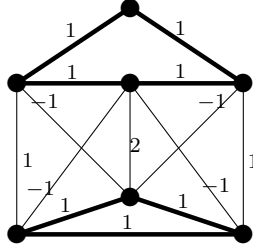


Figure 5: A counterexample for the converse of Lemma 3.

If  $f$  is a  $H_k$ -cordial labeling for a graph  $G$ , in [1] it is defined another labeling  $f^*$  such that  $f^*(e) = k+1-f(e)$  if  $f(e) > 0$ , and  $f^*(e) = -k-1-f(e)$  if  $f(e) < 0$ . It is claimed there that  $f^*$  is also a  $H_k$ -cordial labeling for a graph  $G$ . There exists a simple counterexample to the statement. For the tree shown in Figure 6, the specified labels is a  $H_2$ -cordial labeling, but  $f^*(v) = -5$ .

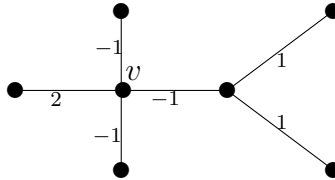


Figure 6: A  $H_2$ -cordial labeling  $f$  for which  $f^*$  is not  $H_2$ -cordial

“Theorem 4.2” [1] states that  $K_n$  is  $H_2$ -cordial, if and only if  $n \equiv 0 \pmod{4}$ . We will show this is not true. We prove the following theorem.

**Theorem 5 .** *The complete graph  $K_n$  is  $H_2$ -cordial, if  $n \equiv 0, 3 \pmod{4}$ , and if  $n \equiv 1 \pmod{4}$  then  $K_n$  is not  $H_2$ -cordial.*

**Proof.** The  $H$ -cordial labelings found in Theorem 3 are also  $H_2$ -cordial labelings. On the other hand if  $n \equiv 2 \pmod{4}$  then  $K_n$  can not have a  $H_2$ -cordial labeling by Lemma 3.  $\square$

The following theorem answers the question of  $H_2$ -cordiality of wheels.

**Theorem 6 .** *Every wheel  $W_n$  has a  $H_2$ -cordial labeling.*

**Proof.** For odd  $n$ , we have an  $H$ -cordial labeling for  $W_n$  by Theorem 4, which is also an  $H_2$ -cordial labeling. Assume that  $n$  is even and the vertex set of  $W_n$  is  $\{v_0, v_1, \dots, v_n\}$ , and  $v_0$  is the central vertex (the vertex with degree  $n$ ). Define  $f(v_i v_{i+1}) = (-1)^i$  where  $1 \leq i \leq n$  and  $v_{n+1} = v_1$ . And define  $f(v_0 v_i) = (-1)^i$  for  $2 \leq i \leq n$ , and  $f(v_0 v_1) = 2$ . It is straight forward to check that  $f$  is a  $H_2$ -cordial labeling for  $W_n$ .  $\square$

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